

Mathematical induction is a method of proving that a statement is true for all positive integers. Here is how it works.

First, show that a statement we will call "statement-one" is true. Second, show that *if* statement-one is true, *then* statement-two must be true and *if* statement-two is true, *then* statement-three must be true, and *if* statement-three is true, *then* statement-four must be true, and so on without end.

Let's label statement-one as S_1 and statement-two as S_2 and so on. Then we can summarize the process described above.

First.

S_1 is true.

Second.

S_1 is true $\implies S_2$ is true,

S_2 is true $\implies S_3$ is true,

S_3 is true $\implies S_4$ is true,

...

The bad news is that it looks like the second part will take some serious time, since apparently one must prove infinitely many implications of the form *if this statement is true, then the next one is true*. That is, infinitely many implications of the form $S_k \implies S_{k+1}$.

The good news is that we can accomplish this in a finite amount of time if we can prove that some statement is true, call it S_1 , and *if any statement- k is true then the next one, statement- $(k+1)$, is must be true, too*. That is, if we can prove S_1 is true and $S_k \implies S_{k+1}$.

The following examples illustrate the proof technique of mathematical induction.

[EX1] Prove: $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$ for $n = 2, 3, 4, \dots$.

Proof.

Let S_n be the statement $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$, $n = 1, 2, 3, \dots$.

Basis. Verify that S_1 is true. S_1 is $1 = \frac{1(1+1)}{2}$. Clearly true.

Inductive step. Show that if S_k is true then S_{k+1} must be true, $k \in \mathbb{Z}^+$.

Suppose S_k is true for some $k \in \mathbb{Z}^+$, i.e.

$$S_k: 1 + 2 + 3 + \dots + k = \frac{k(k+1)}{2}. \text{ [Note: statement } S_k \text{ is known as the induction hypothesis.]}$$

Show that then S_{k+1} must be true, i.e.

$$S_{k+1}: 1 + 2 + 3 + \dots + (k + 1) = \frac{(k+1)((k+1)+1)}{2} = \frac{(k+1)(k+2)}{2}.$$

$$\begin{aligned} \text{LHS} &= 1 + 2 + 3 + \dots + k + (k + 1) \\ &= \frac{k(k+1)}{2} + (k + 1) \text{ [Note: here we have used the hypothesis, statement } S_k \text{.]} \\ &= \frac{k(k+1) + 2(k+1)}{2} \\ &= \frac{(k+1)(k+2)}{2} \\ &= \text{RHS} \end{aligned}$$

Conclusion. If S_k is true, then S_{k+1} is true. Since S_1 is true, S_1, S_2, S_3, \dots are all true.

$\therefore S_n$ is true. $1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$, $n \in \mathbb{Z}^+$

□

For the rest of this, the asterick, *, will indicate when the hypothesis of induction is used.

[EX2] Prove: $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n+1) = \frac{1}{3} n(n+1)(n+2)$ for $n = 1, 2, 3, \dots$.

(J11-BA p. 88 Demonstration 1)

Proof.

Let S_n be the statement $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n+1) = \frac{1}{3} n(n+1)(n+2)$, $n = 1, 2, 3, \dots$.

Basis. S_1 is true because $\text{LHS} = 1 \cdot 2 = 2 = \frac{1(1+1)(1+2)}{3} = \text{RHS}$.

Inductive step.

Suppose S_k is true for some $k \in \mathbb{Z}^+$, i.e.

$$S_k: 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + k(k+1) = \frac{1}{3} k(k+1)(k+2).$$

Show S_{k+1} is true, i.e.

$$S_{k+1}: 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + k(k+1) + (k+1)(k+2) = \frac{1}{3} (k+1)(k+2)(k+3).$$

$$\text{LHS} = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + k(k+1) + (k+1)(k+2)$$

$$= \frac{1}{3} k(k+1)(k+2) + (k+1)(k+2) *$$

$$= \frac{k^3}{3} + k^2 + \frac{2k}{3} + k^2 + 3k + 2$$

$$= \frac{1}{3} (k+1)(k+2)(k+3)$$

$$= \text{RHS}$$

Conclusion. If S_k is true, then S_{k+1} is true. Since S_1 is true, S_1, S_2, S_3, \dots are all true.

$$\therefore 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \dots + n(n+1) = \frac{1}{3} n(n+1)(n+2).$$

□

[EX3] Bernoulli's Inequality (J11-BA p. 89 Demonstration 2)

Bernoulli's Inequality

Let $a > -1$ [Note: says same as $1 + a > 0$]

Thm: $(1 + a)^n \geq 1 + n a$, $n = 0, 1, 2, 3, \dots$

Proof.

Let S_n be the statement $(1 + a)^n \geq 1 + n a$, $n = 0, 1, 2, 3, \dots$

Basis. S_0 is true because $(1 + a)^0 = 1 \geq 1 + 0 a$

Inductive step.

Suppose S_k is true for some $k = 0, 1, 2, \dots$, i.e.

$$S_k: (1 + a)^k \geq 1 + k a$$

Show S_{k+1} is true, i.e.

$$S_{k+1}: (1 + a)^{k+1} \geq 1 + (k + 1) a$$

$$\text{LHS} = (1 + a)^{k+1}$$

$$= (1 + a)^k (1 + a)$$

$$\geq (1 + k a) (1 + a) *$$

$$= 1 + k a + a + k a^2$$

$$\geq 1 + k a + a \quad [\because k a^2 \geq 0]$$

$$= 1 + (k + 1) a$$

$$= \text{RHS}$$

Conclusion. If S_k is true, then S_{k+1} is true. Since S_0 is true, S_0, S_1, S_2, \dots are all true.

$\therefore (1 + a)^n \geq 1 + n a$, $n = 1, 2, 3, \dots$, and $a > -1$.

□

[EX4] Prove: $a(b_1 + b_2 + b_3 + \dots + b_n) = a b_1 + a b_2 + a b_3 + \dots + a b_n$ for $n = 2, 3, 4, \dots$.

Proof.

Let S_n be the statement $a(b_1 + b_2 + b_3 + \dots + b_n) = a b_1 + a b_2 + a b_3 + \dots + a b_n$ for $n = 2, 3, 4, \dots$.

Basis. S_2 is true because $a(b_1 + b_2) = a b_1 + a b_2$

Inductive step.

Suppose S_k is true for some $k \in \mathbb{Z}^+$, i.e.

$$S_k: a(b_1 + b_2 + b_3 + \dots + b_k) = a b_1 + a b_2 + a b_3 + \dots + a b_k$$

Show S_{k+1} is true, i.e.

$$S_{k+1}: a(b_1 + b_2 + b_3 + \dots + b_{k+1}) = a b_1 + a b_2 + a b_3 + \dots + a b_{k+1}$$

$$\text{LHS} = a(b_1 + b_2 + b_3 + \dots + b_k + b_{k+1})$$

$$= a[(b_1 + b_2 + b_3 + \dots + b_k) + b_{k+1}]$$

$$= a(b_1 + b_2 + b_3 + \dots + b_k) + a b_{k+1}$$

$$= a b_1 + a b_2 + a b_3 + \dots + a b_k + a b_{k+1}^*$$

$$= \text{RHS}$$

Conclusion. If S_k is true, then S_{k+1} is true. Since S_2 is true, S_2, S_3, S_4, \dots are all true.

$\therefore a(b_1 + b_2 + b_3 + \dots + b_n) = a b_1 + a b_2 + a b_3 + \dots + a b_n$ for $n = 2, 3, 4, \dots$.

□

[EX5] Prove: $\frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{x^2+1} - \frac{4}{x^4+1} - \dots - \frac{2^{n-1}}{x^{2^{n-1}}+1} = \frac{2^n}{x^{2^n}-1}$, $n \in \mathbb{N}$, $n \geq 2$ (J10 p. 48 #4)

[In book J10 at p. 48 #4, $\frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{x^2+1} - \frac{4}{x^4+1} = \frac{8}{x^8-1}$]

Proof

Basis:

$$S_2: \frac{1}{x-1} - \frac{1}{x+1} = \frac{2}{x^2-1}$$

$$\text{LHS} = \frac{1}{x-1} - \frac{1}{x+1} = \frac{2}{x^2-1} = \text{RHS}$$

Induction:

Suppose S_k is true

$$S_k: \frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{x^2+1} - \dots - \frac{2^{k-1}}{x^{2^{k-1}}+1} = \frac{2^k}{x^{2^k}-1}$$

Show S_{k+1} is true

$$S_{k+1}: \frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{x^2+1} - \dots - \frac{2^{k-1}}{x^{2^{k-1}}+1} - \frac{2^k}{x^{2^k}+1} = \frac{2^{k+1}}{x^{2^{k+1}}-1}$$

$$\begin{aligned} \text{LHS} &= \frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{x^2+1} - \dots - \frac{2^{k-1}}{x^{2^{k-1}}+1} - \frac{2^k}{x^{2^k}+1} \\ &= \frac{2^k}{x^{2^k}-1} - \frac{2^k}{x^{2^k}+1} \quad * \\ &= \frac{2^{k+1}}{x^{2^{k+1}}-1} \end{aligned}$$

=RHS

Since S_k is true implies S_{k+1} is true, and since S_2 is true, all of $S_2, S_3, S_4, S_5, \dots$ are true.

$$\therefore \frac{1}{x-1} - \frac{1}{x+1} - \frac{2}{x^2+1} - \frac{4}{x^4+1} - \dots - \frac{2^{n-1}}{x^{2^{n-1}}+1} = \frac{2^n}{x^{2^n}-1} \quad \square$$

Proof.

Let S_n be the statement

Basis. S_1 is true because

Inductive step.

Suppose S_k is true for some $k \in \mathbb{Z}^+$, i.e.

$$S_k:$$

Show S_{k+1} is true, i.e.

$$S_{k+1}:$$

$$\begin{aligned} \text{LHS} &= \\ &= \text{RHS} \end{aligned}$$

Conclusion. If S_k is true, then S_{k+1} is true. Since S_1 is true, S_1, S_2, S_3, \dots are all true.

\therefore

□